TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 318, Number 2, April 1990

ERGODICITY OF FINITE-ENERGY DIFFUSIONS

TIMOTHY C. WALLSTROM

ABSTRACT. Recently, the existence of a class of diffusion processes with highly singular drift coefficients has been established under the assumption of "finite energy." The drift singularities of these diffusions greatly complicate their ergodicity properties; indeed, they can render the diffusion nonergodic. In this paper, a method is given for estimating the relaxation time of a finite-energy diffusion, when it is ergodic. These results are applied to show that the set of spin- $\frac{1}{2}$ diffusions of stochastic mechanics is uniformly ergodic.

1. INTRODUCTION

In recent years, considerable progress has been made in establishing, in a rigorous mathematical sense, the existence of a class of diffusion processes with highly singular drift coefficients [1, 3, 13, 18]. The primary requirement for these diffusions, which are frequently described as "finite-energy diffusions," is that they satisfy the finite-energy condition:

$$\int_0^T \int_M (b^2(x, t) + b_*^2(x, t))\rho(x, t)\, dx\, dt < \infty. \tag{1}$$

Here, ρ is the probability density on a Riemannian manifold M, and b and b_* are the forward and backward drift coefficients, respectively. We see that under this condition, the drifts are permitted to diverge as $\rho^{-1/2}$ near the zeros of the density, or "nodes."

The reason for considering singularities of this type is that they arise naturally when one considers the time-reversal of a diffusion process, which is also a diffusion process. Indeed, if $\xi(t)$ is a diffusion with forward drift $b(x, t)$ and diffusion constant equal to ν times the metric, then it can be shown that the backward drift b_*, which is defined as the drift of the time-reversed diffusion, is equal to $b - 2\nu\nabla(\rho^{1/2})/\rho^{1/2}$ [14]. If ρ has zeros, we will have singularities of this type.

The purpose of this paper is to study the ergodicity properties of finite-energy diffusions and, in particular, to provide a means for estimating their relaxation

Received by the editors August 10, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A35, 81C20; Secondary 60J60, 58G11, 35P15.
Key words and phrases. Ergodic theory, finite-energy diffusions, singular diffusions, coefficient of ergodicity, spin, stochastic mechanics.

times. The singular drifts introduce a significant new complication, in that they repel the sample paths from regions of low density. In fact, it can be shown that the closest approach of the sample paths is bounded in terms of the integral in (1), so that if the nodal surface separates the manifold, the diffusion is not ergodic [13, 16]. More generally, when the manifold is separated by regions of very low density, the relaxation time can become arbitrarily long. In dealing with nonstationary problems, therefore, it is very useful to have an explicit estimate on the rate at which the ergodic limit is approached.

To provide such an estimate, we reduce the problem to the study of the associated symmetric "osmotic diffusions," and then estimate the lowest nonzero eigenvalue of their generators. This eigenvalue governs the rate at which the ergodic limit is reached. To estimate this eigenvalue, we show how to find constants t and c, both greater than zero, such that $p^t(x, y) > c\rho(y)$ for all x and y. On the one hand, this condition is something which we can prove for finite-energy diffusions, as we show in §4. On the other hand, the eigenvalue λ can be estimated in terms of c, as we show in §3. Indeed, as we also show in §3, λ has an exact expression in terms of Dobrushin's classical coefficient of ergodicity α, and $\alpha > c$.

Before proving the results just described, we establish in §2 our notation, and indicate how to construct the diffusions we will be studying. We provide an application of our results in §5, where we show that the spin-$\frac{1}{2}$ diffusions of stochastic mechanics are uniformly ergodic. This last result will be used in a subsequent paper [17] to prove a conjecture of E. Nelson in the stochastic theory of spin.

2. Construction of the diffusion

First, we review the terminology used in dealing with finite-energy diffusions. We denote our diffusion process by $\xi(t)$. The forward and backward increments are defined by

$$d\xi(t) = \xi(t+dt) - \xi(t), \qquad d_*\xi(t) = \xi(t) - \xi(t-dt),$$

where dt is strictly greater than zero. If $\mathbf{E}^{x,t}[\cdot]$ is the conditional expectation, given that $\xi(t) = x$, then the forward and backward drifts are given by

$$b(x, t) = \lim_{dt\downarrow 0} \mathbf{E}^{x,t}\left[\frac{d\xi(t)}{dt}\right]; \qquad b_*(x, t) = \lim_{dt\downarrow 0} \mathbf{E}^{x,t}\left[\frac{d_*\xi(t)}{dt}\right].$$

This definition is valid in a Euclidean space; for a definition applicable on a Riemannian manifold, see [14].

It is useful to treat finite-energy diffusions in such a way that their time-reversal properties are transparent. To this end, we introduce the so-called osmotic and current velocities, defined respectively by

$$u = \tfrac{1}{2}(b - b_*) \quad \text{and} \quad v = \tfrac{1}{2}(b + b_*).$$

We will assume that the covariance of the process is equal to the metric. The "osmotic velocity," as we have seen, is then equal to

$$\frac{\nabla(\rho^{1/2})}{\rho^{1/2}} = \frac{\nabla\rho}{2\rho} = \frac{1}{2}\nabla\log\rho.$$

The current velocity, in turn, satisfies the continuity equation,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(v\rho). \tag{2}$$

This can be proven by averaging the forward and backward Fokker-Planck equations [14]. We see that the forward drift b is given by $u+v$, and the backward drift by $v-u$. Note also that the finite-energy condition (1) can be expressed in terms of the squares of u and v.

We work on a compact Riemannian manifold M with Riemannian metric g_{ij} and Riemannian connection ∇^i, and consider diffusions $\xi(t)$ defined on a time interval $[0, T]$. Let Z, the nodal set, be defined by $Z = \{(x, t): \rho(x, t) = 0\}$. We assume that the drift coefficients satisfy the finite-energy condition, and that ρ and v satisfy the continuity equation on $M \times [0, T] \setminus Z$.

We construct the diffusions by means of the Itô stochastic integral. Consider the forward and backward stochastic differential equations:

$$d\xi(t) = \beta^i(\xi(t), t)\,dt + dw^i(t), \tag{F}$$

$$d_*\xi(t) = \beta^i_*(\xi(t), t)\,dt + d_*w^i_*(t). \tag{B}$$

Here, β^i and β^i_* are the components of the drift in local coordinates q^i, derived from b and b_* in the usual manner, w (w_*) is the forward (backward) Wiener process with covariance g^{ij}, and d_*f is a backward increment: $d_*f(t) \equiv f(t) - f(t-dt)$. Starting the particle at x at times s, we can solve (F) for $\xi(t)$ by patching together the local solutions of (F) on the manifold $M \times [0, T] \setminus Z$, and stopping the particle when it strikes ∂Z, as in McKean [10, §4.3]. For each $(x, s) \in M \times [0, T] \setminus Z$, we obtain by this procedure a measure $\Pr^{x,s}$. The family of such measures constitute a diffusion for which the strong Markov property, Itô's lemma, and the Girsanov (Cameron-Martin) formula are valid [10]. The transition functions $p(x, s; y, t)$ will be the fundamental solutions to the forward diffusion equations, with Dirichlet boundary conditions on ∂Z. One might question the use of Dirichlet boundary conditions, since this will in general lead to transition probabilities of less than unit mass. In [13], however, Nelson showed that under the assumptions of this section, the particle never strikes the nodes, so boundary conditions on ∂Z are irrelevant, and the transition probabilities have unit mass (see also [16]).

Let path space Ω with typical element ω, be defined as $\prod_{t\in[0,T]} M$, with the product topology, and let $\Pr^{x,s}$ now denote the measure on $(\Omega, \mathscr{B}(\Omega))$ induced by $\xi(t)$. ($\mathscr{B}(\Omega)$ is the Borel algebra of Ω; see [14, §3] for more details.) In this representation $\xi(t)$ is the evaluation functional: $\xi(t)(\omega) = \omega(t)$. Similarly, we define the measure of the process conditioned to have density $\rho(x, 0)$ at $t = 0$ by Pr, and Wiener measure for a particle started at

x at time s by $W^{x,s}$. The forward and backward filtrations are defined, as usual, by

$$\mathscr{P}_t = \sigma\{\xi(s): s \le t\}; \qquad \mathscr{F}_t = \sigma\{\xi(s): s \ge t\}. \tag{3}$$

We could equally well have defined the diffusion using b_* and (B), and we will make use of this fact. If the current velocity v^i is set equal to zero, the drift will be equal to u^i, and either (B) or (F) can be used to define the osmotic diffusions, which are stationary. We use the symbol $\frac{1}{2}\Delta_\rho$ for the osmotic generator:

$$\tfrac{1}{2}\Delta_\rho = \tfrac{1}{2}\Delta + u \cdot \nabla. \tag{4}$$

If U is a Borel subset of M, define the stopping time $T_U(\xi)$ as $\inf\{t \ge 0: \xi(t) \in U\}$.

3. An estimate on the relaxation constant

In §4, we will show that the relaxation of the solution to the diffusion equation in the $L^2(\rho)$ norm depends only on the symmetric part of the generator, which in turn corresponds to the stationary diffusion with invariant density ρ and current velocity equal to zero. For this reason, we restrict ourselves in this section to selfadjoint operators.

Let A be the generator of a stationary Markovian process on a manifold M. Let ρ be the invariant density, $p^t(x, y)$ the transition probability density, and $\lambda = \inf\{\|Af\|: \mathbf{E}f = 0,\ \|f\| = 1\}$, where the norms $\|\cdot\|$ refer to the weighted Hilbert space $L^2(M, \rho\, dx)$, and $\mathbf{E}f$ is defined as $\int f(x)\rho(x)\, dx$. The quantity λ is sometimes called the relaxation constant of the process; it is inversely proportional to the relaxation time. Assume that A is selfadjoint in $L^2(\rho)$. We then have the following result.

Theorem 1. *Suppose there exist* $t > 0$ *and* $0 < c < 1$, *such that*

$$p^t(x, y) \ge c\rho(y) \qquad (x \in M,\ y \in M). \tag{5}$$

Then

$$\lambda \ge \frac{1}{t} \log \frac{1}{1 - c}. \tag{6}$$

Proof. First, note that if $f \ge 0$, $f \in L^2(\rho)$, and $\mathbf{E}f = 1$, then $P^t f = c + g$, where $g \ge 0$ and $\mathbf{E}g = 1 - c$. (We get strict equality since $(1, P^t f) = (P^t 1, f) = \|f\|_1$.)

Now let $f \in L^2(\rho)$ be an eigenfunction of A in $1^\perp$, so that $\mathbf{E}f = (f, 1) = 0$. Since A is real and selfadjoint, we can assume without loss of generality that f is real, and since $\|f\|_1 < \|f\|_2$, we can normalize f so that $\|f\|_1 = 1$. We will show that

$$\|P^t f\|_1 \le 1 - c,$$

which will imply the estimate on λ above.

Let $f = f_1 - f_2$, where $f_1 = f \wedge 0$, and $f_2 = -f \wedge 0$. Then $\mathbf{E}f_1 = \mathbf{E}f_2 = \frac{1}{2}$. By the above,

$$P^t(2f_1) = c + g_1 \quad \text{and} \quad P^t(2f_2) = c + g_2,$$

where $g_1 \geq 0$, $g_2 \geq 0$, and $\mathbf{E}g_1 = \mathbf{E}g_2 = 1 - c$. Therefore,

$$\|P^t f\|_1 \leq \tfrac{1}{2}(\|g_1\|_1 + \|g_2\|_1) = 1 - c. \quad \square$$

This result is actually a corollary of classical results on the coefficient of ergodicity, due to Dobrushin [5, 6]. (See also [8, 9, 15].) Indeed, we will show below that

$$\lambda = \frac{1}{t} \log \frac{1}{1-\alpha}, \tag{7}$$

where α is the coefficient of ergodicity of P^t. Dobrushin provides several equivalent expressions for α; the following remarkable expression, equation 1.5‴ in [5], was suggested by remarks of Kolmogorov:

$$\alpha = \inf_{x_1, x_2} \tilde{\alpha}(p^t(x_1, \cdot), p^t(x_2, \cdot)), \tag{8}$$

where $\tilde{\alpha}(\mu_1, \mu_2)$ is the mass of the largest measure minorizing both μ_1 and μ_2. It is clear from this expression that $\alpha \geq c$ (as it must be), and in most cases, we will have $\alpha > c$.

We have given an independent proof of Theorem 1, because it is simple and direct, does not presuppose familiarity with Dobrushin's work, and provides the condition we will need for our subsequent results. We now sketch how Dobrushin's work can be used to obtain the exact result (7).

In our proof, we estimated λ by estimating

$$M(P) = \sup\{\|P^t f\|_1 : \mathbf{E}f = 0,\ \|f\|_1 = 1,\ f \text{ real}\}. \tag{9}$$

We have consider the transition function as it acts on the right, on $L^1(\rho)$. Dobrushin considers P as it acts on the left, on the space of measures:

$$P\colon \mu(\cdot) \mapsto \int_M \mu(dx) P(x, \cdot).$$

He then considers the norm of the transition function on the subspace of measures of zero mass,

$$N(P) = \sup\{\|\mu P\|_v : \mu(M) = 0,\ \|\mu\|_v = 1\}, \tag{10}$$

where $\|\cdot\|_v$ is the total variation norm. He shows that $\alpha = 1 - N(P)$.

For a general stochastic operator P (that is, P such that $P1 = 1$), there seems to be no simple relationship between $M(P)$ and $N(P)$. But for P^t selfadjoint on $L^2(\rho)$ and $p^t(x, y)$ continuous in x and y, these quantities are identical, as we will show. First, note that if the measure $d\mu$ is defined as $f\rho\,dx$, where f satisfies the constraints in (9), then μ satisfies the constraints in (10). Next, note that for P^t selfadjoint, $\rho(x)p^t(x, y) = p^t(y, x)\rho(y)$. From this it can easily be shown that $\|P^t f\|_1 = \|\mu P\|_v$. We can therefore identify

$M(P)$ with $N(P)$, provided the same $N(P)$ is attained when we also contrain μ to have a density of the form $f\rho\,dx$. This is essentially the requirement that μ have a density with respect to Lebesgue measure, and will not affect $N(P)$ provided $p^t(x, y)$ is continuous in x and y.

I would like to thank the referee for encouraging me to investigate possible connections between my Theorem 1 and existing results on the eigenvalue gap.

4. Ergodicity of finite-energy diffusions

First, we will show that the relaxation of a finite-energy diffusion depends only on the density, and thus only on the associated osmotic diffusions. This reduces our problem to the study of the stationary osmotic diffusions for a density ρ. We then show that for these diffusions, the condition $p^t(x, y) > c\rho(y)$ can be established under the assumption that the nodal set Z, now defined simply as $\{x: \rho(x) = 0\}$, has a neighborhood $U(Z)$ satisfying the two following conditions: (i) $M \setminus U(Z)$ is connected, and (ii) $\Delta\rho > 0$ on $\overline{U(Z)}$. The first condition is very close to assuming that $M_+ \equiv M \setminus Z$ is connected, which is necessary. The second condition will be used to show that the sample paths leave the neighborhood of Z with uniform promptness.

Let $\xi(t)$ be a finite-energy diffusion on the manifold M, whose drift is C^∞ away from the nodes. Let $\mathbf{E}^{x,t}$ be the conditional expectation given that the particle is at x at time t. Define the Markovian propagators $P(t, s)$ and $P_*(t, s)$ by

$$P(t, s)f(x) = \mathbf{E}^{x,t}[f(\xi(s))] \qquad (t < s),$$
$$P_*(t, s)f(x) = \mathbf{E}^{x,t}[f(\xi(s))] \qquad (t > s).$$

For fixed s, these generate solutions to the forward and backward diffusion equations, respectively. If f and g are scalars on M, define the weighted inner product $(\cdot, \cdot)_t$ by

$$(f, g)_t = \int_M f(x)g(x)\rho(x, t)\,dx. \tag{11}$$

If an argument in the inner product is time dependent, it is assumed to have the value indicated by the bracket subscript.

Define $f(x, t) = P_*(t, s)f(x)$, and let us calculate $(d/dt)\|f\|_t^2$, using probabilistic methods. Let the stochastic process $F(t)$ be defined as $f(\xi(t), t)$. By Itô's lemma,

$$d_*(F^2) = 2F d_*F - (d_*F)^2, \qquad d_*F = \nabla F \cdot d_*w_*,$$

and therefore

$$d_*(F^2) = 2F\nabla F \cdot d_*w_* - (\nabla F)^2\,dt.$$

Taking expectations, we get

$$\frac{d}{dt}\|f\|_t^2 = -\|\nabla f\|_t^2. \tag{12}$$

Note that $\|\nabla f\|_t^2 = -(f, \Delta_\rho f)_t$, and that Δ_ρ is independent of the current velocity v. Therefore, to study the relaxation of $f(x, t)$ it is sufficient to consider only the associated osmotic processes.

Theorem 2. *Let ρ be a smooth probability density on a compact manifold M, let Z denote the nodal set of ρ, and let $\xi(t)$ be the corresponding osmotic diffusion. Assume there exists an open neighborhood $U(Z)$ of Z such that $A \equiv M\setminus U(Z)$ is connected and $\Delta\rho > 0$ on $\overline{U(Z)}$. Then there exist $c > 0$ and $t > 0$ such that*

$$p^t(x, y) \geq c\rho(y) \qquad (x \in M_+,\ y \in M_+). \tag{13}$$

Proof. The proof will be divided into three steps. Before beginning the proof, however, we prove a lemma which will be necessary for Step 2. Recall that T_A is defined as the stopping time for the first entrance into A (§2).

Lemma. *Under the assumptions of the theorem, there exists a time t_A, depending only on $K = \inf_{x\in U(Z)} \Delta\rho(x)$ and $\varepsilon = \sup_{x\in U(Z)} \rho(x)$, such that $\Pr^x\{T_A < t_A\} \geq 1/2$ for all x in $U(Z)$.*

Proof. In the proof of this lemma, we will use $\xi(t)$ to denote the osmotic diffusion started at $x \in U(Z)$ at time 0. By Itô's lemma,

$$d\rho(\xi(t)) = \tfrac{1}{2}\Delta\rho\, dt + 2\rho u^2 dt + 2\rho u \cdot dw.$$

This implies that

$$\begin{aligned}\rho(\xi(t\wedge T_A)) - \rho(x) &= \int_0^{t\wedge T_A} (\tfrac{1}{2}\Delta\rho + 2\rho u^2)(\xi(s))\, ds + \int_0^{t\wedge T_A} 2\rho u \cdot dw \\ &\geq K(t\wedge T_A) + \int_0^{t\wedge T_A} 2\rho u\cdot dw.\end{aligned}$$

Taking expectations, we find that

$$\mathbf{E}\rho(\xi(t\wedge T_A)) \geq Kt \Pr^x\{T_A \geq t\}.$$

Since $\mathbf{E}\rho(\xi(t\wedge T_A)) \leq \varepsilon$,

$$\Pr^x\{T_A \geq t\} \leq \varepsilon/Kt.$$

The lemma follows for $t_A = 2\varepsilon/K$. □

We now commence with the proof of the theorem.

Step 1. $\exists c' > 0$ such that

$$p^t(x, y) \geq c' \qquad (t \in [t_A, 2t_A],\ x \in A,\ y \in A). \tag{14}$$

This is a simple consequence of the fact that $p^t(x, y)$ is jointly continuous in t, x, and y, and by the strong maximum principle greater than zero on the compact set in question.

Step 2.

$$p^{2t_A}(x, y) \geq c'/2 \qquad (x \in M_+,\ y \in A). \tag{15}$$

If $x \in A$, use Step 1. If $x \in M_+ \setminus A$,

$$\begin{aligned}\int_A p^{2t_A}(x, y)\,dy &= \Pr^x[\xi(2t_A) \in A] \\ &= \Pr^x[\Pr^{\xi(T_A)}[\xi(2t_A - T_A) \in A]] \\ &\geq \Pr^x[\Pr^{\xi(T_A)}[\xi(2t_A - T_A) \in A];\ T_A \leq t_A] \\ &= \int_{\partial A \times [0, t_A]} \Pr^z[\xi(2t_A - r) \in A]\,d\mu(z, r),\end{aligned}$$

where the product measure μ is defined on $M \times [0, t_A]$ by

$$\mu(V \times W) = \Pr^x[\xi(T_A) \in V,\ T_A \in W].$$

But $\mu(\partial A \times [0, t_A]) \geq 1/2$, by the lemma, and

$$\Pr^z[\xi(2t_A - r) \in A] = \int_A p^{2t_A - r}(z, y)\,dy \geq c' \int_A dy,$$

when $0 \leq r \leq t_A$. So

$$p^{2t_A}(x, y) \geq c'/2.$$

Step 3. $\exists c > 0$ such that

$$p^{4t_A}(x, y) > c\rho(y) \qquad (x \in M_+,\ y \in M_+). \tag{16}$$

The backward transition function $p_*^t(x, y) = p^t(y, x)$ so [14, equation 6.2]

$$\rho(x)p^t(x, y) = p^t(y, x)\rho(y). \tag{17}$$

From this and Step 2, we get

$$p^{2t_A}(x, y) > \frac{c'}{2c_0}\rho(y) \qquad (x \in A,\ y \in M_+),$$

where $c_0 \equiv \sup_{x \in M_+} \rho(x)$. But if $x \in M_+$, $y \in M_+$, the Chapman-Kolmogorov equations give

$$\begin{aligned}p^{4t_A}(x, y) &\geq \int_A p^{2t_A}(x, z)p^{2t_A}(z, y)\,dz \\ &\geq \frac{c'^2 \int_A dz}{4c_0}\rho(y) = c\rho(y). \qquad \square\end{aligned}$$

5. Uniform relaxation of spin-$\frac{1}{2}$ diffusions

We now apply these results to the stochastic mechanical theory of spin. Stochastic mechanics is an alternative interpretation of quantum mechanics in which the evolution of the quantum-mechanical probability density is understood in terms of classical diffusion theory. The fundamental equation of nonrelativistic quantum mechanics for particles without spin is the Schrödinger equation,

$$i\partial\psi/\partial t = (-\tfrac{1}{2}\Delta + V)\psi, \tag{18}$$

where $\psi \in L^2(M)$. If we average ψ^* times this equation with $-\psi$ times its complex conjugate, we get the continuity equation (2), with $\rho = |\psi|^2$ and $v = \operatorname{Im}(\nabla\psi/\psi)$. We can therefore associate to this density evolution a finite-energy diffusion with forward drift $v + u$ and backward drift $v - u$. The covariance of the diffusion is usually taken as $\nu = \hbar/m$, where $\hbar$ is Planck's constant and m is the particle mass. We have tacitly chosen units in which $\hbar = m = 1$, so $\nu = 1$.

The theory of spin in stochastic mechanics was developed by Dankel [4]. His work was based on earlier work of Bopp and Haag [2], who gave quantum mechanical equations for a rigidly charged sphere in an electromagnetic field. Dankel associated with solutions to the Bopp-Haag equation a diffusion on $\mathbb{R}^3 \times SO(3)$, working by analogy to the earlier work of Nelson [11, 12] on the Schrödinger equation. Bopp and Haag had shown that as the sphere's moment of inertia I goes to zero, the solutions of the Bopp-Haag equations would converge to the solution of an I-independent equation. This equation is a $(2s+1)$-fold copy of the Pauli equation, which is, in turn, the accepted nonrelativistic description of particles with spin. This is the basis of the expectation that Bopp and Haag's model will provide a stochastic interpretation for particles with spin.

The assumption of ergodic behavior for the SO(3) component of the Bopp-Haag-Dankel diffusions has been fundamental to attempts to understand the $I \to 0$ limit probabilistically, since it offers the prospect of approximating orientation dependent quantities, such as the spatial drift, by their averages over the orientational density. The effective diffusion coefficient of the orientational component goes as I^{-1}, so the ergodic limit will be approached arbitrarily rapidly as I goes to zero [4, 14, 17]. (Note: the ergodicity proof given in Theorem 13 of [4] is not correct.)

All spin-$\frac{1}{2}$ wavefunctions have nodes, so any attempt to prove ergodicity runs into the obstacles discussed in the introduction. All spin-$\frac{1}{2}$ wavefunctions, however, also satisfy the hypotheses of Theorem 2, so the methods of this paper suffice to establish the existence of a relaxation constant for any individual wavefunction. By consulting the proof of Theorem 2 to see which factors contribute to the constants t and c, we will show that the set of individual relaxation constants has a strictly positive infimum.

The Hilbert space associated with spin diffusions is $L^2(SU(2))$. We recall that

$$L^2(SU(2)) = \bigoplus_{n=0}^{\infty} \mathscr{H}^{n/2},$$

where each $\mathscr{H}^s$ is an eigenspace of the Laplacian with eigenvalue $s(s+1)$. The stochastic theory takes $\mathscr{H}^s$ as the spin-s state space. It may appear that by considering $L^2(SU(2))$, we have abandoned our intention to consider diffusions on the classical configuration space SO(3). Any spin-s wavefunction, however, will lead to a density and drift coefficients which are single-valued on SO(3), and it is from these we build our spin-s diffusions.

It is computationally useful to exploit the diffeomorphism between SU(2) and $S^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$ in order to represent functions on SU(2) as functions of

$$x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4.$$

The manifold SO(3) is then diffeomorphic to P^3, projective three-space, obtained from S^3 by identifying x with $-x$. See [7] for additional information.

Theorem 3. *The relaxation constants of the spin-$\frac{1}{2}$ diffusions have a strictly positive infimum.*

Proof. As shown above, it will be sufficient to consider the osmotic diffusions associated with the spin-$\frac{1}{2}$ wavefunctions. It will be sufficient, furthermore, to consider densities of the form

$$\rho = a^2 x_1^2 + b^2 x_2^2 \qquad (a, b \in \mathbb{R},\ a^2 \geq b^2). \tag{19}$$

Indeed, any ψ in $\mathscr{H}^{1/2}$ can be written, after suitable rotations, as $\psi = ax_1 + i(bx_1 + cx_2)$, where a, b, and c are real. The corresponding density $\rho = a^2x_1^2 + (bx_1 + cx_2)^2$. A rotation in the x_1-x_2 plane can always be found to take this to $\rho = a^2x_1^2 + b^2x_2^2$, where a and b have been redefined. In general, a continuum of wavefunctions will have this density, but the simplest representative, and the one we will use for computational purposes, is $\psi = ax_1 + ibx_2$. We will assume further that $\|\psi\| = 1$, or equivalently, that $a^2 + b^2 = 4$.

Referring back to the proof of Theorem 2, we find that

$$c = \frac{c'^2 \int_A dz}{4c_0} \quad \text{and} \quad t = 4t_A.$$

We will divide the densities into two cases and obtain uniform constants for each case. The larger t and the smaller c will suffice as uniform constants for all spin-$\frac{1}{2}$ diffusions.

Let $b_0^2 = \varepsilon/2$, for some small ε, and consider the following two cases:

Case 1. $b^2 \leq b_0^2$. Take $U^{(1)}(Z) = \{x : x_1^2 < \varepsilon/8\}$.

Case 2. $b^2 > b_0^2$. Take $U^{(2)}(Z) = \{x : x_1^2 + x_2^2 < \varepsilon/4\}$.

Note that $\rho(x) < \varepsilon$ on $U^{(i)}(Z)$ for all ρ in Case (i).

It is not difficult to show that $\frac{1}{2}\Delta\rho = 1 - \rho$ for ρ of the form (19). We may therefore choose $t_A = \varepsilon/(2(1-\varepsilon))$ for both cases, by the lemma. Since $\|\rho\|_\infty \leq 4$ for all ψ in $\mathscr{H}^{1/2}$, we may choose $c_0 = 4$. We need only show, therefore, that $c' > 0$ can be chosen uniformly for the ρ in each case.

In the proof of Theorem 2, we showed merely that $c' > 0$; in this proof we use the Girsanov formula to derive an estimate for c' that is uniform for the densities in each case. The Girsanov formula states that the Radon-Nikodym derivative of $\Pr^x$ with respect to Wiener measure W^x is equal to

$$z(t) = \exp\left\{ \int_0^t u(w(s)) \cdot dw - \frac{1}{2} \int_0^t u^2(w(s))\, ds \right\}. \tag{20}$$

If R is defined by $\rho = e^{2R}$, Itô's lemma applied to $R(w(s))$ gives

$$u \cdot dw = dR - \tfrac{1}{2}\nabla \cdot u\, ds.$$

Therefore

$$z(t) = \left(\frac{\rho(w(t))}{\rho(w(0))}\right)^{1/2} \exp \int_0^t U(w(s))\, ds, \tag{21}$$

where $U(x) = -\frac{1}{2}\Delta(\rho^{1/2})/\rho^{1/2}$.

Consider the osmotic diffusions in Case 1. Let

$$B = M \setminus \{x: x_1^2 < \varepsilon/16\},$$
$$D_t = \{\omega: \omega(t) \in V,\ T_{M\setminus B}(\omega) \geq t,\ \omega \text{ is continuous on } [0, t]\},$$

where V is a Borel subset of $A^{(1)} = M \setminus U^{(1)}(Z)$. Then B is a set whose interior properly contains $A^{(1)}$, and D_t is the set of continuous paths which never leave B in the time interval $[0, t]$, and which are in $V \subset A^{(1)}$ at time t. Let

$$U_{\max}(t) = \sup\{|U(x)|: x \in B,\ \rho \text{ in Case } 1\},$$
$$\rho_{\min}(t) = \inf\{\rho(x): x \in B,\ \rho \text{ in Case } 1\},$$
$$z_{\min}(t) = \sqrt{\frac{\rho_{\min}}{\rho_0}} \exp(-tU_{\max}).$$

Clearly $z_{\min}(t)$ is greater than zero, since $U_{\max} < \infty$ and $\rho_{\min} > 0$, as can easily be shown by explicit calculation. Now $W^x(D_t) = \int_V q_B^t(x, y)\, dy$, where $q_B^t(x, y)$ is the transition probability density for Brownian motion started at x and stopped on leaving B, and similarly, $\Pr^x(D_t) = \int_V p_B^t(x, y)\, dy$, where $p_B^t(x, y)$ is the analogous transition probability density for the osmotic diffusion $\xi(t)$. Let

$$c_{\min} = \inf\{q_B^t(x, y): (t, x, y) \in [t_A, 2t_A] \times A \times A\}. \tag{22}$$

Then, as in Step 1 of Theorem 2, $c_{\min} > 0$, and $p_B^t(x, y) > c_{\min} z_{\min}(2t_A)$ on $[t_A, 2t_A] \times A \times A$. Since $p^t(x, y) > p_B^t(x, y)$, (14) holds uniformly for all ρ in Case 1 if

$$c' = c_{\min} z_{\min}(2t_A).$$

A similar procedure gives a uniform constant in Case 2.

Finally, we must verify that the sets $A^{(1)}$ and $A^{(2)}$ are connected subsets of $\mathrm{SO}(3) \cong P^3$ for every wavefunction in $\mathscr{H}^{1/2}$. Consider first the set $A^{(1)}$. Let x and x' be representatives in $\mathbb{R}^4$ of two points in $P^3 \setminus \{x: x_1^2 > \varepsilon/8\}$. The only possible difficulty in drawing a line in $A^{(1)}$ connecting these points arises when $x_1 x_1' < 0$, since at first glance it might appear that the line will have to pass through $x_1 = 0$. We can, however, represent the first point by $-x_1$ instead, and this difficulty is then eliminated. For $A^{(2)}$, there are no such problems: if we wish to move from a positive to a negative value of x_1 or x_2, or vice versa, we simply hold the absolute value of the other coordinate high enough to avoid the nodal surface. □

6. Discussion

It would be interesting to know whether the condition that $\Delta\rho$ is strictly greater than zero on a neighborhood of the nodes can be removed. As noted above, this condition is only needed for the lemma, where it is used to ensure that the particle leaves the nodal neighborhood promptly. But since $u = \frac{1}{2}\nabla \log \rho$ diverges as $\rho \to 0$ for any continuous ρ, one might expect that the particle will always leave the nodal neighborhood promptly.

The result in §5 will be used in a subsequent paper [17] to prove Nelson's conjecture that the spatial projections of the spin-$\frac{1}{2}$ Bopp-Haag-Dankel diffusions converge, as the moment of inertia I goes to 0, to a Markovian limit process. In fact, for this latter proof, Theorem 3 is the only result needed which depends on the specific value of the spin. It is interesting to note therefore that Theorem 3 is false for integral spin: the nodes of any real wavefunction with integral spin will disconnect SO(3). Nevertheless, there may be important subspaces of $\mathscr{H}^s$, such as invariant representation spaces for SU(2), for which an analogous result holds.

Acknowledgments. This paper is part of the author's Princeton University Ph.D. thesis. The author would like to thank his thesis advisor, Professor Edward Nelson, for his encouragement and interest in the work, and for his careful review of the manuscript.

References

1. Ph. Blanchard and S. Golin, *Diffusion processes with singular drift fields*, Comm. Math. Phys. **109** (1987), 421–435.
2. Fritz Bopp and Rudolf Haag, *Über die Möglichkeit von Spinmodellen*, Z. Naturforsch. **5a** (1950), 644–653.
3. Eric A. Carlen, *Conservative diffusions*, Comm. Math. Phys. **94** (1984), 293–315.
4. Thaddeus George Dankel, Jr., *Mechanics on manifolds and the incorporation of spin into Nelson's stochastic mechanics*, Arch. Rational Mech. Anal. **37** (1970), 192–222.
5. R. L. Dobrushin, *Central limit theorem for nonstationary Markov chains*. I, Theory Probab. Appl. **1** (1956), 65–79.
6. ——, *Central limit theorem for nonstationary Markov chains*. II, Theory Probab. Appl. **1** (1956), 329–383.
7. William G. Faris, *Spin correlation in stochastic mechanics*, Found. Phys. **12** (1982), 1–26.
8. J. Hajnal, *Weak ergodicity in non-homogeneous Markov chains*, Proc. Cambridge Philos. Soc. **54** (1958), 233–246.
9. M. Iosifescu, *Finite Markov processes and their applications*, Wiley, 1980.
10. H. P. McKean, Jr., *Stochastic integrals*, Probab. Math. Statist., vol. 5, Academic Press, 1969.
11. Edward Nelson, *Derivation of the Schrödinger equation from Newtonian mechanics*, Phys. Rev. **150** (1966), 1079–1085.
12. ——, *Dynamical theories of Brownian motion*, Princeton Univ. Press, Princeton, N.J., 1967.

13. ———, *Critical diffusions*, Séminaire de Probabilités XIX 1983/84 (J. Azéma and M. Yor, editors), Lecture Notes in Math., vol. 1123, Springer-Verlag, 1985, pp. 1–11.

14. ———, *Quantum fluctuations*, Princeton Univ. Press, Princeton, N.J., 1985.

15. E. Seneta, *Coefficients of ergodicity: structure and applications*, Adv. Appl. Probab. **11** (1979), 576–590.

16. Timothy C. Wallstrom, *A finite-energy bound on the approach of a diffusion to zeros of its density*, Proc. Amer. Math. Soc. **108** (1990), 839–843.

17. ———, *The stochastic mechanics of the Pauli equation*, Trans. Amer. Math. Soc. **318** (1990), 749–762.

18. W. A. Zheng, *Tightness results for laws of diffusion processes. Application to stochastic mechanics*, Ann. Inst. H. Poincaré **21** (1985), 103–124.

Department of Physics, Princeton University, Princeton, New Jersey 08544

Current address: 21 East Camino Real, Arcadia, California 91006